An introduction to o-minimality

Notes by Tom Foster and Marcello Mamino on lectures given by
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1 Introduction

Throughout these notes definable means definable with parameters, unless otherwise stated.

Definition 1. The structure $\mathcal{R} := \langle R, <, \ldots \rangle$ is said to be *o-minimal* if the only definable subsets of R are finite unions of intervals and points, ie. those sets definable with just the ordering.

Example 1. The real field $\overline{\mathbb{R}} = \langle \mathbb{R}, <, +, \cdot, 0, 1 \rangle$ is o-minimal. This is a consequence of Tarski's theorem that $\overline{\mathbb{R}}$ admits quantifier elimination. In fact we may conclude something stronger. Let $\phi(x)$ be a formula in the language of $\overline{\mathbb{R}}$ with parameters from $\mathbb{R}$. Writing $\phi(x)$ as $\bigwedge_i \bigvee_j p_{ij}(x) \square_{ij} 0$, where $\square_{ij} \in \{<, =\}$ and $p_{ij} \in \mathbb{R}[X]$, we see that we have an upper bound, namely $\Sigma_{ij} \deg(p_{ij})$, on the set of points needed to describe $\phi(\overline{\mathbb{R}})$ which is independent of the parameters used in ϕ.

This is a special case of the following general property of definable families in o-minimal structures over the reals:

Theorem 1. *Let $\mathbb{R}^*$ be any o-minimal expansion of $\langle \mathbb{R}, < \rangle$ and suppose that $\phi(\bar{x}, \bar{y})$, where $\bar{x}$ and $\bar{y}$ are tuples of length n and m respectively, is a (parameter-free) formula in the language of $\mathbb{R}^*$. Then there exists $N \in \mathbb{N}$ such that for all $\bar{a} \in \mathbb{R}^n$ the set $\{\bar{b} \in \mathbb{R}^m : \mathbb{R}^* \models \phi(\bar{a}, \bar{b})\}$ has at most N connected components. Furthermore each connected component is definable.*

Remark 1. Using this theorem we can deduce that o-minimality is preserved under elementary equivalence.

1.1 o-minimality via model completeness

Definition 2. We say that a first-order theory T is *model complete* if, modulo T, every formula is equivalent to an existential formula. When we say that a structure is model complete we mean that its theory is model complete.

Theorem 2 (Robinson's test for model completeness)**.** *Let T be a first-order theory. Suppose that T has the following property:*

> *For any $\mathcal{M}, \mathcal{N} \vDash T$ such that $\mathcal{M} \subseteq \mathcal{N}$, and any quantifier-free formula $\phi(\bar{v})$ with parameters in $\mathcal{M}$, if $\mathcal{N} \vDash \exists \bar{v} \phi(\bar{v})$ then $\mathcal{M} \vDash \exists \bar{v} \phi(\bar{v})$.*

Then T is model-complete.

Theorem 3. *Let $\mathbb{R}^*$ be an expansion of $\langle \mathbb{R}, < \rangle$ and suppose that:*

1. *for all $n \in \mathbb{N}$ every quantifier-free definable subset of $\mathbb{R}^n$ has finitely many connected components.*
2. *the theory of $\mathbb{R}^*$ is model-complete.*

Then our structure $\mathbb{R}^*$ is o-minimal.

Proof. Let X be a definable subset of $\mathbb{R}^*$. By the model-completeness of $\mathbb{R}^*$ X is defined by an existential formula $\exists \bar{y} \phi(\bar{y}, x)$, i.e. $\phi(\bar{y}, x)$ is quantifier-free. The set, Y say, defined by $\phi(\bar{y}, x)$ must have finitely many connected components. X is the image of Y under the projection map $\pi : \langle \bar{y}, x \rangle \to x$ so must have finitely many connected components also. The only connected subsets of $\mathbb{R}$ are intervals and points so X is of the desired form. □

Henceforth, $\tilde{\mathbb{R}} := \langle \overline{\mathbb{R}}, \mathcal{F} \rangle$ where $\mathcal{F}$ is some collection of functions from $\mathbb{R}^n$ to $\mathbb{R}$, for various n.

Definition 3. We call a term in the language of $\tilde{\mathbb{R}}$ *simple* if has the form $\tau(\bar{x}) = p(x_1, \ldots, x_n, F_1(\bar{x}^{(1)}), \ldots, F_s(\bar{x}^{(s)}))$, where $\bar{x} = (x_1, \ldots, x_n)$, $p(x_1, \ldots, x_n, y_1, \ldots, y_s) \in \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_s]$, $F_1, \ldots F_s \in \mathcal{F}$ and the $\bar{x}^{(i)}$ are subsequences of $\bar{x}$ of suitable length.

For example, if $\mathcal{F} = \{\exp\}$ then a simple term is of the from

$$p(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}).$$

Theorem 4. *For $\tilde{\mathbb{R}} := \langle \overline{\mathbb{R}}, \mathcal{F} \rangle$ as above, every existential formula in the language of $\tilde{\mathbb{R}}$ is equivalent, modulo the theory of $\tilde{\mathbb{R}}$, to one of the form $\exists \bar{y}(\tau(\bar{y}, \bar{x}) = 0)$ where τ is a simple term.*

Proof. Exercise.

Hint: $f(g(\bar{x})) = 0$ iff $\exists y(y = g(\bar{x}) \wedge f(y) = 0)$ iff $\exists y(f(y)^2 + (g(\bar{x}) - y)^2 = 0)$ □

2 Khovanski's Theorem

Many early attempts to prove o-minimality were based on Khovanski's finiteness theorem.

Notation. For $U \subseteq \mathbb{R}^n$ open and $m \in \mathbb{N} \cup \{\infty\}$, $\mathcal{C}^m(U)$ denotes the set of all functions $f : U \to \mathbb{R}$ that are m-times continuously differentiable on U.

Definition 4. Let $U \subseteq \mathbb{R}$. A sequence $f_1, \ldots, f_r \in \mathcal{C}^1(U)$ is called a *Pfaffian chain* if there exist polynomials $p_i(x, y_1, \ldots, y_i) \in \mathbb{R}[x, y_1, \ldots, y_i]$, for $i = 1 \ldots r$, such that for all $x \in U$

$$\begin{aligned} f_1'(x) &= p_1(x, f_1(x)) \\ f_2'(x) &= p_2(x, f_1(x), f_2(x)) \\ &\vdots \\ f_r'(x) &= p_r(x, f_1(x), \ldots, f_r(x)). \end{aligned}$$

More generally, if $U \subseteq \mathbb{R}^n$, we require polynomials $p_{ij}(x_1, \ldots, x_n, y_1, \ldots, y_i)$ for $j = 1, \ldots, n$, $i = 1, \ldots, r$ such that

$$\frac{\partial f_i}{\partial x_j}(x_1, \ldots, x_n) = p_{ij}(x_1, \ldots, x_n, f(x_1, \ldots, x_n), \ldots, f_i(x_1, \ldots, x_n)).$$

We say that a function $f \in \mathcal{C}^1(U)$ is Pfaffian if $f(\bar{x}) = q(\bar{x}, f_1(\bar{x}), \ldots, f_r(\bar{x}))$ for some polynomial q and some Pfaffian chain $f_1, \ldots, f_r$ on U.

The following facts about Pfaffian functions are left as exercises.

Exercise 1.

1. If $f \in \mathcal{C}^1(U)$ is Pfaffian then $f \in \mathcal{C}^\infty(U)$.
2. The class of Pfaffian functions includes many well-known and commonly occurring functions, for example exp on $\mathbb{R}$, log on $(0, \infty)$, sin on $(0, \pi)$ and all rational functions on their domains.
3. The class of Pfaffian functions is closed under many operations, for example, paying close attention to domains, composition, differentiation, integration and extracting roots via the implicit function theorem.

Theorem 5 (Khovanski's finiteness theorem)**.** *Let $U \in \mathbb{R}^m$ be an open box and suppose that $f \in \mathcal{C}^\infty(U)$ is Pfaffian. Then there exists a natural number n, depending only upon the degrees of the p_{ij}'s and the q as in definition 4, such that $Z(f) := \{\bar{x} \in U : f(\bar{x}) = 0\}$ has at most N connected components.*

Exercise 2. Prove the following special case of theorem 5: Let $U \subseteq \mathbb{R}$ be an interval, and let $f \in \mathcal{C}^\infty(U) \setminus 0$ satisfy $f'(x) = p(x, f(x))$ for some $p(x, y) \in \mathbb{R}[x, y]$. Prove that f has only finitely many simple zeroes. Find a bound n in terms of $deg(p)$. Now use Sard's Theorem to prove that f has only finitely many zeroes (approximate all zeroes by simple zeroes of a slightly different function). If you can extend this result to $q(x, f(x))$, for $q(x, y) \in \mathbb{R}[x, y]$, then you are well on the way to seeing the general case.

Exercise 3. Prove that $\sin : \mathbb{R} \to \mathbb{R}$ is not Pfaffian but $\sin : (0, \pi) \to \mathbb{R}$ is. What about $\sin : (0, 2\pi) \to \mathbb{R}$?

Now let $f_1, \ldots, f_r$ be a Pfaffian chain on some open box $U \subseteq \mathbb{R}^n$. Suppose we would like to show that $\tilde{\mathbb{R}} := \langle \bar{\mathbb{R}}, f_1, \ldots, f_r \rangle$ is o-minimal (set $f_i(\bar{x}) = 0$ if $\bar{x} \notin U$). By theorems 3 and 5 it is sufficient that $\tilde{\mathbb{R}}$ is model complete.

In fact, we only know the model completeness of $\tilde{\mathbb{R}}$ in a few cases:

1. $n = r = 1$, $U = \mathbb{R}$ and $f_1 = \exp : x \mapsto e^x$.

2. arbitrary n, r but on bounded U such that $f_1, \ldots f_r$ have extensions, satisfying the same differential equations, to some open box V with $\bar{U} \subseteq V$.

However the o-minimality of $\tilde{\mathbb{R}}$ *is* known in general, but by different methods.

3 Real analytic functions

Let $U \subseteq \mathbb{R}$ be an open interval and let $f \in \mathcal{C}^\infty(U)$. For $x_0 \in U$, we form the (formal) Taylor series of f around x_0:

$$T^f_{x_0}(x) = \Sigma_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i \tag{1}$$

Then f is called *analytic* at x_0 if there exists $r > 0$ (with $(x_0 - r, x_0 + r) \subseteq U$) such that for all $y \in (x_0 - r, x_0 + r)$ the series of real numbers $T^f_{x_0}(y)$ converges with sum $f(y)$. We say that f is *analytic on* U if it is analytic at x_0 for all $x_0 \in U$.

Exercise 4. Let $f \in \mathcal{C}^\infty(U)$ where U is an open interval in $\mathbb{R}$. Prove that the following are equivalent:

1. f is analytic on U.

2. for each compact set $K \subseteq U$ there exists a constant $C > 0$ such that for all $i \geq 0$ and for all $x \in K$, $|f^{(i)}(x)| \leq C^{i+1} i!$

The definition of an analytic function generalizes easily to many variables once we have introduced multi-index notation. The result in exercise 4 also generalizes. One just replaces x, x_0 by tuples, and i by a multi-index:-

Notation. An element $\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle \in \mathbb{N}^n$ is called a *multi-index*. We set

$$\begin{aligned}
|\alpha| &:= \alpha_1 + \ldots + \alpha_n \\
\alpha! &:= \alpha_1! \ldots \alpha_n! \\
f^{(\alpha)} &:= \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \\
\bar{x}^\alpha &:= x_1^{\alpha_1} \ldots x_n^{\alpha_n} \quad \text{where } \bar{x} = (x_1, \ldots, x_n)
\end{aligned}$$

We write $\mathcal{C}^\omega(U)$ for the set of analytic functions on U, where U is an open subset of $\mathbb{R}^n$ for some n.

Fact 1. *1. The class of analytic functions is closed under composition, differentiation, integration and extracting roots via the implicit function theorem.*

2. Every Pfaffian function is analytic on its domain.

3. *If f is analytic on a connected open neighbourhood U of the point x_0 and $f^{(\alpha)}(x_0) = 0$ for all $\alpha \in \mathbb{N}^n$ then $f \equiv 0$ on U.*

Definition 5. For $n \in \mathbb{N} \setminus \{0\}$ we define to $\mathcal{G}_n$ to be the set of restrictions to $[-1,1]^n$ of analytic functions defined on open neighbourhoods of $[-1,1]^n$, ie. $f : [-1,1]^n \to \mathbb{R} \in \mathcal{G}_n$ iff there exists an open box V with $[-1,1]^n \subseteq V$ and $g \in \mathcal{C}^\omega(V)$ such that $f = g \restriction_{[-1,1]^n}$. We let $\mathcal{G}_0 := \mathbb{R}$ and now define $\mathbb{R}_{an} := \langle \bar{\mathbb{R}}, \bigcup_{n \geq 0} \mathcal{G} \rangle$ (we set our functions to zero outside the unit box).

We have the following theorems about the structure $\mathbb{R}_{an}$.

Theorem 6 (Gabrielov (1969))**.** *$\mathbb{R}_{an}$ is model complete.*

Theorem 7 (Denef, van den Dreis [1])**.** *$\langle \mathbb{R}_{an}, D \rangle$ admits quantifier elimination, where*

$$D : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto \begin{cases} \frac{x}{y} & |x| \leq |y| \neq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

4 Some local theory of analytic functions

Definition 6. Let

- I be the interval $[-1,1]$ in $\mathbb{R}$;
- L_{an} be the language which has, for each $n \geq 0$ and each $f \in \mathcal{G}_n$ (see definition 5), a function symbol of arity n.
- the language L_{an}^D be $\{D(\cdot,\cdot), <\} \cup L_{an}$;
- and, finally, $\mathcal{I}$ be the L_{an}^D-structure defined on the domain I by interpreting any function symbol in L_{an}^D as the analytic function it names, D as

$$D(x,y) = \begin{cases} \frac{x}{y} & \text{if } |x| \leq |y| \text{ and } 0 < |y| \\ 0 & \text{otherwise,} \end{cases}$$

  and $<$ in the usual way (moreover we will use $>$, $\leq$ and $\geq$ as shorthands for the corresponding definitions in terms of $<$ and $=$).

We are interested in proving the following

Theorem 8. *The structure $\mathcal{I}$ has quantifier elimination.*

This implies theorem 7.

Proof that theorem 8 implies theorem 7. Let L be the language of our $\langle \mathbb{R}_{an}, D \rangle$. Observe that $\langle \mathbb{R}_{an}, D \rangle$ can be interpreted in $\mathcal{I}$ by mapping each $x \in \mathbb{R}$ to the unique pair $(a,b) \in I^2$ such that $\frac{a}{b} = x$ and either a or b (or both) is 1. Moreover each quantifier free L_{an}^D-formula is, ipso facto, an L-formula. Hence, suppose we want to eliminate the quantifiers in some L-formula $\phi(x_1,\dots,x_n)$: we first consider its interpretation as an L_{an}^D-formula $\psi(a_1, b_1, \ldots, a_n, b_n)$, then, by theorem 8, we have an equivalent quantifier free L_{an}^D-formula $\psi'(a_1, b_1, \ldots, a_n, b_n)$, and, finally, we notice that ϕ is equivalent in $\langle \mathbb{R}_{an}, D \rangle$ to an appropriate boolean combination of the formulæ$|x_i| \leq 1$ and instances of ψ' with a_i and b_i replaced respectively by either x_i and 1 or $D(1, x_i)$ and 1. □

Notice that, even if the symbols D and $<$ in the language L_{an}^D are definable in L_{an}, we are not able to achieve the QE without them.

Definition and facts

First of all we will recall some definitions and facts about the real analytic functions.

Definition 7. The ring $\mathcal{O}_x^n$ of *germs of analytic functions at x* is defined as the quotient of the ring of functions from $\mathbb{R}^n$ to $\mathbb{R}$ which are analytic on a neighbourhood of 0 by the equivalence relation $\sim_x$ defined by

$$f \sim_x g \quad \leftrightarrow \quad \exists \epsilon > 0 \, \forall y \, |x - y| < \epsilon \rightarrow f(y) = g(y)$$

i.e. identifying functions which are pointwise equal in some neighbourhood of x.

Of course, $\mathcal{O}_0^n$ is isomorphic to the ring $\mathbb{R}\langle x_1, \ldots, x_n \rangle$ of the power series in n variables convergent in some neighbourhood of 0. This observation enables us to state the following

Fact 2. *The ring $\mathbb{R}[[x_1, \ldots, x_n]]$ of formal power series in n variables is faithfully flat over the ring $\mathcal{O}_0^n$. In particular, a linear equation with coefficients in $\mathcal{O}_0^n$ has a solution in $\mathcal{O}_0^n$ if and only if it has one in $\mathbb{R}[[x_1, \ldots, x_n]]$.*

We will need the Weierstrass preparation theorem. Let us first introduce the following definition.

Definition 8. Let $f \in \mathcal{O}_0^{n+1}$. We say that f is *regular* of degree d in the variable x_{n+1} if f has non-vanishing dth derivative in x_{n+1} at 0 and zero ith derivative in x_{n+1} at 0 for all $i < d$. We say that f is regular if it is regular of degree d for some d.

Theorem 9 (Weierstrass preparation theorem). *Let $f(x_1, \ldots, x_{n+1}) \in \mathcal{O}_0^{n+1}$ be a germ of analytic functions which is regular of degree d in x_{n+1}. Then*

$$f = u \cdot p$$

where u is a unit of $\mathcal{O}_0^{n+1}$, and p is a monic polynomial in the indeterminate x_{n+1} of degree d with coefficients in $\mathcal{O}_0^n$.

Overview of the proof

In the proof we will often deal with terms and formulæ in two sets of variables. We will write them as $\phi(\bar{x}, \bar{y})$, where $\bar{x}$ represents an m-tuple of variables and $\bar{y}$ represents an n-tuple of variables. Moreover we will call *simple* a formula or term in which no instance of D is applied to a term involving any of the variables in $\bar{y}$. The symbols i and j will be used to denote multiindices in $\mathbb{N}^n$, the norm $|i|$ of a multiindex is the maximum of its components.

We are now ready to prove the QE theorem. By a well known observation, all we have to prove is that for any quantifier free L_{an}^D-formula $\phi(\bar{x}, y)$ exists a quantifier free L_{an}^D-formula $\psi(\bar{x})$ such that

$$I \models \exists y \phi(\bar{x}, y) \leftrightarrow \psi(\bar{x})$$

(where $\bar{x}$ represents any number of free variables).

Our strategy is to find a new quantifier free L_{an}^D-formula ϕ' which is simple and satisfies

$$I \models \exists y \phi(\bar{x}, y) \leftrightarrow \exists \bar{y} \phi'(\bar{x}, \bar{y})$$

(thus eliminating the Ds at the expense of an increase in the number of existential quantifiers).

Now we have a string of n existential quantifiers to eliminate, and suppose *by induction* that we can eliminate $n-1$ existential quantifiers (in front of a simple formula). Our goal is to reduce the formula $\exists \bar{y} \phi'(\bar{x}, \bar{y})$ to a boolean combination of simple formulæ each of which has at most $n-1$ existential quantifiers, thus concluding by induction.

More specifically, *assuming ϕ' to be just an L_{an}-formula* (a detail which has no influence on the argument overall), we will prove that for any $(\bar{p}, \bar{q}) \in I^m \times I^n$ there exists an $\epsilon_{(\bar{p},\bar{q})} > 0$ and a simple quantifier free L_{an}^D-formula $\phi''_{(\bar{p},\bar{q})}(\bar{x}, z_1, \ldots, z_{n-1})$ such that

$$I \models \exists \bar{y} \phi'(\bar{p} + \epsilon_{(\bar{p},\bar{q})} \bar{x}, \bar{q} + \epsilon_{(\bar{p},\bar{q})} \bar{y}) \leftrightarrow \exists \bar{z} \phi''_{(\bar{p},\bar{q})}(\bar{x}, \bar{z})$$

This means that for any point $(\bar{p}, \bar{q}) \in I^m \times I^n$ we can get a simple formula which is equivalent to our one *restricted to a neighbourhood of that point* and has one quantifier less. Finally, we will conclude that, by compactness of $I^m \times I^n$, $\exists \bar{y} \phi'(\bar{x}, \bar{y})$ is equivalent to a boolean combination of just a finite number of such formulæ.

Of course, our assumption that ϕ' is an L_{an}-formula can be easily fulfilled by replacing each instance of D with a dummy variable and then substituting the original terms back in ϕ''.

Details of the proof

The first step is easily concluded by replacing any (annoying) occurrence of D with its definition

$$I \models z = D(x, y) \leftrightarrow (x^2 \leq y^2 \rightarrow x = zy) \wedge ((x^2 > y^2 \vee y = 0) \rightarrow z = 0)$$

(in which all functions, i.e. square and product, are in L_{an}) existentially quantifying the dummy variable z.

For the second step, we will work locally around $(\bar{p}, \bar{q})$. To simplify notation, we suppose, without loss of generality, that $(\bar{p}, \bar{q}) = 0$; moreover we make the assumption that each atomic subformula of ϕ' is of the form $t(\bar{x}, \bar{y}) > 0$ with t a term in the language L_{an}. Our aim, now, is to find a suitable value for $\epsilon = \epsilon_{(\bar{p},\bar{q})}$ as described above, thus filling in the missing part of the argument.

Of course, if an existentially quantified variable, say y_n, occurs in ϕ' just polynomially (i.e. if each term of ϕ' is a polynomial in the indeterminate y_n whose coefficients are analytic functions in the remaining variables) then we can eliminate it by Tarski's theorem (possibly multiplying each polynomial by a small enough constant in order to constrain the coefficients in I).

Now, the Weierstrass preparation theorem provides us with a tool for making a variable occur just polynomially, at least locally. So, were each term in ϕ' regular in y_n , we could rewrite it locally around $(\bar{p}, \bar{q})$, which is around 0 by our assumption, so that y_n occurs just polynomially. More specifically, if we can write each term in ϕ' as $t = u \cdot f$ locally at 0, with suitable $f \in \mathcal{O}_0^{m+n-1}[y_n]$ and unit $u \in \mathcal{O}_0^{m+n}$, supposing without loss of generality $u > 0$ in a neighbourhood of 0, then we have for a small enough $\epsilon > 0$

$$I \vDash t(\epsilon\bar{x}, \epsilon\bar{y}) > 0 \leftrightarrow f(\epsilon\bar{x}, \epsilon\bar{y}) > 0$$

which is sufficient to conclude by taking a value of ϵ small enough to work for every term in ϕ', and invoking Tarski's theorem as per the previous observation.

Hence, the rest of this section will be devoted to refining this argument in order to deal with the general case (when some of the terms may not be regular in y_{n+1} at $(\bar{p}, \bar{q})$, which we assumed to be 0).

Lemma 1. *For any $f(\bar{x}, \bar{y}) \in \mathcal{O}_0^{m+n}$ there is a positive integer d such that $f(\bar{x}, \bar{y})$ can be written as*

$$f(\bar{x}, \bar{y}) = \sum_{|i|<d} a_i(\bar{x})\bar{y}^i u_i(\bar{x}, \bar{y})$$

with $a_i \in \mathcal{O}_0^m$ and u_i units of $\mathcal{O}_0^{m+n}$.

Proof. By induction on n.
Case $n = 1$. Write the power series $f(\bar{x}, y) = \sum_k a_k(\bar{x})y^k$ with $a_k(\bar{x}) \in \mathbb{R}\langle x_1, \ldots, x_m\rangle$. Since $\mathbb{R}[[x_1, \ldots, x_m]]$ is noetherian, there is an integer d such that the ideal generated by all the $a_k(\bar{x})$ is already generated by the $a_k(\bar{x})$ with $k < d$. Hence we can find the units u_k in $\mathbb{R}[[x_1, \ldots, x_m, y]]$, and, by fact 2, even in $\mathbb{R}\langle x_1, \ldots, x_m, y\rangle$.
Case $n > 1$ By induction we can write

$$f(\bar{x}, \bar{y}) = \sum_{|i|<d} a_i(x, y_1)(y_2, \ldots, y_n)^i u_i(\bar{x}, \bar{y})$$

from which we can conclude applying the case $n = 1$ to each of the $a_i(x, y_1)$. □

We are going to analyze an atomic sub formula of ϕ': $t(\bar{x}, \bar{y}) > 0$. By the lemma we can rewrite t as $\sum_{|i|<d} a_i(\bar{x})\bar{y}^i u_i(\bar{x}, \bar{y})$ in some neighbourhood of 0. Unless specified otherwise, each formula from now on is to be intended for $(\bar{x}, \bar{y})$ in *that* neighbourhood of 0.

Let $\mu_j(\bar{x})$ be the formula

$$a_j(\bar{x}) \neq 0 \wedge \left(\bigwedge_{|i|<d} |a_j(\bar{x})| \geq |a_i(\bar{x})| \right)$$

we observe that by shrinking the neighbourhood if neccessary we get

$$t(\bar{x}, \bar{y}) > 0 \leftrightarrow \bigvee_{|j|<d} (\mu_j(\bar{x}) \wedge t(\bar{x}, \bar{y}) > 0) \tag{*}$$

(take j such that $|a_j(\bar{x})|$ is maximal, *etc*). We now focus on a single j, analyzing further the formula $\mu_j(\bar{x}) \wedge t(\bar{x}, \bar{y}) > 0$.

Introducing the new variables v_i, we define

$$\tilde{t}(\bar{x}, \bar{v}, \bar{y}) = \bar{y}^j u_j(\bar{x}, \bar{y}) + \sum_{\substack{|i|<d \\ i \neq j}} v_i \bar{y}^i u_i(\bar{x}, \bar{y})$$

so, whenever $\mu_j(\bar{x})$ is true, we have

$$t(\bar{x}, \bar{y}) = a_j(\bar{x}) \tilde{t}(\bar{x}, \bar{v'}(\bar{x}), \bar{y})$$

where $v_i'(\bar{x}) = D(a_i(\bar{x}), a_j(\bar{x}))$.

We claim that after a suitable transformation of the variables $\bar{y}$, the Weierstrass preparation theorem can be applied to $\tilde{t}_{\bar{c}}$ defined by:

$$\tilde{t}_c(\bar{x}, \bar{v}, \bar{y}) = \tilde{t}(\bar{x}, \bar{c} + \bar{v}, \bar{y})$$

for any $\bar{c} \in I^{(\{1,\ldots,d\} \setminus \{j\})^n}$.

The transformation $\bar{\lambda}(\bar{y})$ is defined by

$$\lambda_k(\bar{y}) = \begin{cases} y_k + y_n^{d^{n-k}} & \text{if } k \neq n \\ y_n & \text{if } k = n \end{cases}$$

and it has inverse

$$\omega_k(\bar{y}) = \begin{cases} y_k - y_n^{d^{n-k}} & \text{if } k \neq n \\ y_n & \text{if } k = n \end{cases}$$

our claim is that $\tilde{t}_{\bar{c}}(\bar{x}, \bar{v}, \bar{\lambda}(\bar{y}))$ is regular in y_n. Indeed, consider the lexicographically smallest multiindex ι such that $\bar{y}^\iota$ has non null coefficient in the power series of $\tilde{t}_{\bar{c}}(0, 0, \bar{y})$: since u_i is a unit for every i, $|\iota| < d$ (in fact, ι either is j or lexicographically smaller than j with $c_\iota \neq 0$). Now, the series of $\tilde{t}_{\bar{c}}(0, 0, \bar{\lambda}(0, \ldots, 0, y_n))$ has order precisely $\sum_{n=1,\ldots,n} \iota_k d^{n-k}$, since for any i lexicographically larger than ι

$$\sum_{n=1,\ldots,n} \iota_k d^{n-k} < \sum_{n=1,\ldots,n} i_k d^{n-k}$$

so it is regular.

By the Weierstrass preparation theorem

$$\tilde{t}_{\bar{c}}(\bar{x}, \bar{v}, \bar{\lambda}(\bar{y})) = u_{\bar{c}}(\bar{x}, \bar{v}, \bar{y})\, f_{\bar{c}}(\bar{x}, \bar{v}, \bar{y})$$

with $f_{\bar{c}} \in \mathcal{O}_0^{\text{whatever}}[y_n]$ and unit $u_{\bar{c}} > 0$ (and whatever $= m + n + d^n - 2$). Hence

$$\tilde{t}_{\bar{c}}(\bar{x}, \bar{v}, \bar{y}) = u_{\bar{c}}(\bar{x}, \bar{v}, \bar{\omega}(\bar{y}))\, f_{\bar{c}}(\bar{x}, \bar{v}, \bar{\omega}(\bar{y}))$$

and, by taking a function in each germ, the equality holds as well in some neighborhood $I_{\bar{c}}$ of 0.

At this point, for some $\epsilon_{\bar{c}} > 0$ we have the equivalence

$$t(\bar{x}, \bar{\lambda}(\bar{y})) > 0 \leftrightarrow a_j(\bar{x}) f_{\bar{c}}(\bar{x}, \bar{v'}(\bar{x}) - \bar{c}, \bar{y}) > 0$$

whenever $\mu_j(\bar{x})$ is true and the absolute values of $v_i'(\bar{x}) - c_i$, x_k and y_l (for all i, k and l) are all smaller than $\epsilon_{\bar{c}}$. Observing that y_n occurs just polynomially in $a_j(\bar{x}) f_{\bar{c}}(\bar{x}, \bar{v'}(\bar{x}) - \bar{c}, \bar{y})$, possibly using the already mentioned trick of multiplying each term by a small enough constant, we may consider the right hand side of the equivalence *a simple L_{an}^D-formula in which y_n occurs just polynomially.*

Now the dependency on $\bar{c}$ can be easily eliminated by compactness of $I^{\{1,\ldots,d\}^n \setminus \{j\}}$ and observing that $|v_i'(\bar{x}) - c_i| < \epsilon_{\bar{c}}$ is (equivalent to) an L_{an}^D-formula. More precisely, for some $\epsilon > 0$ and some finite set C of multiindices, whenever $\mu_j(\bar{x})$ is true and the absolute values of x_k and y_l (for all k and l) are all smaller than ϵ, $t(\bar{x}, \bar{\lambda}(\bar{y})) > 0$ is equivalent to

$$\bigwedge_{\bar{c} \in C} |v_i'(\bar{x}) - c_i| < \epsilon_{\bar{c}} \to f_{\bar{c}}(\bar{x}, \bar{v'}(\bar{x}) - \bar{c}, \bar{y}) > 0$$

which, again, is a simple L_{an}^D-formula in which y_n occurs just polynomially.

In the end, substituting the former in (*) for each j, for each atomic sub formula $t(\bar{x}, \bar{\lambda}(\bar{y})) > 0$ of $\phi'(\bar{x}, \bar{\lambda}(\bar{y}))$ we have a simple L_{an}^D-formula in which y_n occurs just polynomially equivalent to it in a neighbourhood of 0. Substituting them and invoking Tarski's theorem we accomplish our goal of locally eliminating one existential quantifier.

5 The full exponential function

We aim to show that the real field expanded by the full exponential function, which we will denote by $\mathbb{R}_{\exp}$, is model-complete. As remarked in section 2 we can then conclude that $\mathbb{R}_{\exp}$ is o-minimal.

Definition 9. An expansion $\tilde{\mathbb{R}}$ of the real field is said to be *polynomially bounded* if for all definable functions $f : \mathbb{R} \to \mathbb{R}$ there exists a natural number N such that $|f(x)| \leq x^N$ for all sufficiently large x.

Remark 2. One might think that the full exponential function is somehow definable in $\mathbb{R}_{an}$. If this were the case then the o-minimality of $\mathbb{R}_{\exp}$ would follow directly from the o-minimality of $\mathbb{R}_{an}$. However this is not the case. Indeed, in [4] van den Dries proves that the structure $\mathbb{R}_{an}$ is polynomially bounded, and hence cannot define the full exponential function. There are, however, some global analytic functions definable in $\mathbb{R}_{an}$.

Example 2. $\sin \restriction_{(-\pi/2,\pi/2)}$ and $\cos \restriction_{(-\pi/2,\pi/2)}$ are both definable in $\mathbb{R}_{an}$. Consequently $\tan \restriction_{(-\pi/2,\pi/2)}$ is definable in $\mathbb{R}_{an}$. We may define $\tan^{-1}$ on $\mathbb{R}$ by saying that it is the inverse map to tan. So $\langle \overline{\mathbb{R}}, \tan^{-1} \rangle$ is o-minimal.

As remarked above $\mathbb{R}_{an}$ is polynomially bounded and so $\langle \overline{\mathbb{R}}, \exp \restriction_{[-1,1]} \rangle$ is polynomially bounded. We now develop some theory of polynomially bounded o-minimal expansions of the real field. From now on, $\tilde{\mathbb{R}}$ will denote such a structure.

Theorem 10 (Miller [2])**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be definable in $\tilde{\mathbb{R}}$. Then there exists a unique $\alpha \in \mathbb{R}$ such that $\frac{f(x)}{x^\alpha}$ converges to a non-zero finite limit as $x \to \infty$.*

Exercise 5. With f and α as above, prove that α and $x \mapsto x^\alpha : (0, \infty) \to \mathbb{R}$ are definable over the same set of parameters as f. Furthermore, prove that the set of all such α forms a subfield of $\mathbb{R}$; this is known as the *field of exponents* of $\tilde{\mathbb{R}}$.

From now we will assume that $\tilde{\mathbb{R}}$ has field of exponents $\mathbb{Q}$. This is the case for $\mathbb{R}_{an}$ [4] and hence for $\langle \overline{\mathbb{R}}, \exp \restriction_{[-1,1]} \rangle$. Let $\tilde{T}$ be the theory of $\tilde{\mathbb{R}}$ and let $\tilde{\mathcal{L}}$ be it's language. Let $\mathcal{M} = \langle M, <, \ldots \rangle \models \tilde{T}$. We define

$$\mu(\mathcal{M}) := \{x \in M : \forall q \in \mathbb{Q}_{>0}\ |x| < q\}$$
$$Fin(\mathcal{M}) := \{x \in M : \exists q \in \mathbb{Q}_{>0} \text{ s.t. } |x| < q\}.$$

$Fin(\mathcal{M})$ is a subring of $\mathcal{M}$ and $\mu(\mathcal{M})$ is the unique maximal ideal of $Fin(\mathcal{M})$. In [5] van den Dries shows that the field $K := Fin(\mathcal{M})/\mu(\mathcal{M})$ can be expanded to an $\tilde{\mathcal{L}}$ structure $\tilde{K}$ such that, up to isomorphism, $\tilde{K}$ can be elementarily embedded in $\mathcal{M}$. In fact the embedding can be chosen so that each $\mu(\mathcal{M})$ equivalence class of $Fin(\mathcal{M})$ contains exactly one element of $\tilde{K}$.

Now $Fin(\mathcal{M})\backslash\mu(\mathcal{M})$ is a subgroup of the multiplicative group of $\mathcal{M}$, $\mathcal{M}\backslash\{0\}$. We let Γ be the quotient

$$^{(\mathcal{M}\backslash\{0\})}\big/_{(Fin(\mathcal{M})\backslash\mu(\mathcal{M}))},$$

which we will write additively, and we let $v : \mathcal{M} \setminus \{0\} \to \Gamma$ be the quotient map. Observe that since $\mathcal{M}$ is real closed Γ is divisible and hence a $\mathbb{Q}$-vector space.

Recall from [3] that the definable closure operator is a pregeometry in o-minimal structures so we have a notion of dimension. Furthermore, since o-minimal expansions of groups have definable Skolem functions, the dimension of our structure $\mathcal{M}$ is given by:

$\inf\{n : \exists n \text{ elements of } M \text{ which generate } \mathcal{M} \text{ under 0-definable Skolem functions}\}$.

Theorem 11 (The valuation inequaliy [6]). *With $\mathcal{M}$, $\tilde{K}$ and Γ as above, if* $\dim(\mathcal{M})$ *is finite then*

$$\dim(\mathcal{M}) \geq \dim(\tilde{K}) + \dim_{\mathbb{Q}}(\Gamma).$$

We also have a relative version of the valuation inequality: let $K_1, K_2 \models \tilde{T}$ such that $K_1 \preccurlyeq K_2$. Let Γ_1, Γ_2 denote the value groups of K_1, K_2 and $\tilde{K}_1, \tilde{K}_2$ be their residue fields. If $\dim_{K_1}(K_2)$ is finite then

$$\dim_{K_1}(K_2) \geq dim_{\mathbb{Q}}(\Gamma_1/\Gamma_2) + \dim_{\tilde{K}_1}(\tilde{K}_2).$$

For our purposes we will only use the fact that $\dim_{K_1}(K_2) \geq dim_{\mathbb{Q}}(\Gamma_2/\Gamma_1)$. Unravelling the statement, this means: if $\dim_{K_1}(K_2) = p$, and $a_1, \ldots a_{p+1} \in K_2$ then there exists $n_1, \ldots, n_{p+1} \in \mathbb{Z}$, not all zero, and $c \in K_1$ such that $ca_1^{n_1} \ldots a_{p+1}^{n_{p+1}} \in Fin(K_2) \setminus \mu(K_2)$.

5.1 Proof of model completeness of $\mathbb{R}_{\text{exp}}$

Let T_{exp} denote the theory of $\mathbb{R}_{\text{exp}}$. Let $\mathcal{M}_1, \mathcal{M}_2 \models T_{\text{exp}}$ and suppose that $\mathcal{M}_1 \subseteq \mathcal{M}_2$. We claim that $\mathcal{M}_1 \preccurlyeq_{\exists_1} \mathcal{M}_2$ and hence that T_{exp} is model complete.

Sketch proof of claim. Let $F(x_1, \ldots, x_n)$ be a simple term with parameters in $\mathcal{M}_1$ and assume that there exists $a_1, \ldots, a_n \in \mathcal{M}_2$ such that $F(a_1, \ldots, a_n) = 0$. By Robinson's test and theorem 4 it is sufficient to prove that there exists $b_1, \ldots, b_n \in \mathcal{M}_1$ such that $F(b_1, \ldots, b_n) = 0$.

Step 1 $F(x_1, \ldots, x_n)$ is of the form $P(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$ where $P(\bar{x}, \bar{y}) \in \mathcal{M}_1[\bar{x}, \bar{y}]$. We construct $P_1, \ldots, P_n \in \mathcal{M}_1[\bar{x}, \bar{y}]$ and $\bar{a}' \in {\mathcal{M}_2}^n$ such that $\bar{a}'$ is a non-singular zero of the system of equations

$$P_i(\bar{x}, e^{\bar{x}}) \quad \text{(i=1,\ldots,n)} \tag{2}$$

and a zero of

$$P(\bar{x}, e^{\bar{x}}). \tag{3}$$

Step 2 We prove that if $\bar{a}'$ is a non-singular of the system (2) which is bounded between elements of $\mathcal{M}_1$ (ie. there exists $b \in \mathcal{M}_1$ such that $|a_i'| < b$ for $i = 1, \ldots n$) then in fact $\bar{a}'$ lies in $\mathcal{M}_1$.

Step 3 It now remains to prove that any non-singular zero of the system (2) in $\mathcal{M}_2$ is bounded between elements of $\mathcal{M}_1$. This is where we will use the valuation inequaltiy and the model completeness of $\langle \overline{\mathbb{R}}, \exp \restriction_{[0,1]} \rangle$. Let $\langle b_1, \ldots, b_n \rangle$ be a non-singular solution of (2). We consider the structures $\mathcal{M}_1^* := \langle \overline{\mathcal{M}}_1, \exp \restriction_{[} -1, 1] \rangle$ and $\mathcal{M}_2^* := \langle \overline{\mathcal{M}}_2, \exp \restriction_{[} -1, 1] \rangle$, where the bar indicates the ordered field structure only. Let K_2 be the elementary substructure of $\mathcal{M}_2^*$ generated by $b_1, \ldots, b_n, e^{b_1}, \ldots, e^{b_n}$ and $\mathcal{M}_1^*$. Then

$\dim_{\mathcal{M}_1^*} K_2 \leq n$, therefore by the valuation inequality, for each $r = 1, \ldots, n$ there exists $c \in \mathcal{M}_1^*$ and $a_0, \ldots, a_n \in \mathbb{Z}$ not all zero such that

$$cb_r^{a_0} \prod_{i=1}^{n} e^{a_i b_i} \in Fin(K_2) \setminus \mu(K_2).$$

By some combinatorial arguments we can now find $a'_1, \ldots, a'_n \in \mathbb{Z}$ not all zero and $\alpha \in \mathcal{M}_1^*$ such that $-\alpha \leq \sum_{i=1}^{n} a'_i b_i \leq \alpha$. Now by a linear transformation of variables we may suppose that $-\alpha \leq b_n \leq \alpha$. Now use an inductive argument.

□

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